

Surfaces in Sol_3 space foliated by circles

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Abstract

In this paper we study surfaces foliated by a uniparametric family of circles in the homogeneous space Sol_3 . We prove that there do not exist such surfaces with zero mean curvature or with zero Gaussian curvature. We extend this study considering surfaces foliated by geodesics, equidistant lines or horocycles in totally geodesic planes and we classify all such surfaces under the assumption of minimality or flatness.

1 Statement of the results

In Euclidean space \mathbb{E}^3 , the plane and the catenoid are the only minimal rotational surfaces. These surfaces are foliated by a uniparametric family of coaxial circles. We consider now a more general context and we ask for those minimal surfaces foliated by a uniparametric family of circles in parallel planes, called cyclic surfaces [2, 8]. A straightforward computation shows that, besides the rotational surfaces, the surface must be one of the classical family of minimal surfaces discovered by Riemann [8, 9]. Similarly, the only cyclic surfaces in \mathbb{E}^3 with zero Gaussian curvature are rotational surfaces, or generalized cones [4].

The motivation of the present paper is to consider cyclic surfaces in Sol_3 space and to classify them under the assumptions of minimality and flatness. This space is a simply connected homogeneous 3-manifold and it belongs to one of the eight models of the geometry of Thurston [11]. As a Riemannian manifold, the space Sol_3 can be represented by \mathbb{R}^3 equipped with the metric

$$\langle \cdot, \cdot \rangle = e^{2z} dx^2 + e^{-2z} dy^2 + dz^2,$$

where (x, y, z) are the canonical coordinates of \mathbb{R}^3 . The space Sol_3 is a Lie group with the group operation

$$(x, y, z) * (x', y', z') = (x + e^{-z} x', y + e^z y', z + z'),$$

and the metric $\langle \cdot, \cdot \rangle$ is left-invariant. Recently, there is an activity on the study of submanifolds in Sol_3 , see for example [1, 3, 5, 6, 7, 10].

Recall that in Sol_3 there is not a group of rotations as in the Euclidean space \mathbb{E}^3 , that is, a uniparametric continuous group of direct isometries that leave pointwise a geodesic. However, a rotational surface in \mathbb{E}^3 is also characterized as a surface foliated by the uniparametric circles C_s all contained in planes P_s whose centers form a straight line orthogonal to each P_s . In Sol_3 we have all these concepts, such as geodesic, totally geodesic surface, and distance. Under this viewpoint, we carry the next definition:

Definition 1.1. *A circle in Sol_3 is a curve contained in a totally geodesic surface P and equidistant from a fixed point of P , called the center of the circle.*

We point out that the distance in the circle is computed with the induced metric on P by Sol_3 , which agrees with the one induced on the curve by the metric of the ambient space because P is totally geodesic.

We recall that in Sol_3 the totally geodesic surfaces are the vertical planes P_s of equation $x = s$ and the planes Q_s of equation $y = s$, where $s \in \mathbb{R}$. Here we use the notion of 'vertical' and 'horizontal' in its usual affine concept of the set \mathbb{R}^3 , where the space Sol_3 is modeled.

By analogy with what happens in Euclidean space, we give the next definition.

Definition 1.2. *A cyclic surface in Sol_3 is a surface foliated by a uniparametric family of circles contained in the planes P_s (resp. contained in the planes Q_s). If the centers of these circles form a geodesic line orthogonal to each plane P_s (resp. each plane Q_s), then we say that the surface is rotational.*

Motivated by the results of Euclidean space \mathbb{E}^3 , we ask for those cyclic surfaces in Sol_3 that are minimal (zero mean curvature) or flat (zero Gaussian curvature). For example, we ask about the existence of rotational surfaces or surfaces with the same role as Riemann minimal examples.

Because the study of cyclic surfaces foliated by circles in the planes P_s is analogous to the ones foliated by circles in the planes Q_s , we will assume in this paper that the cyclic surfaces are always foliated by circles in the totally geodesic planes of type P_s .

In this sense, we obtain:

Theorem 1.3. *There are no minimal cyclic surfaces in Sol_3 .*

Theorem 1.4. *There are no flat cyclic surfaces in Sol_3 .*

Both theorems will be proved in Section 3. On the other hand, recall that the planes P_s are isometric to the hyperbolic plane \mathbb{H}^2 and the circles in P_s are curves with constant curvature κ , $\kappa > 1$. However, in the hyperbolic plane there are other curves with constant curvature κ , but $0 \leq \kappa \leq 1$. Indeed, these curves are geodesics, equidistant lines and horocycles. Therefore, we ask for minimal or flat cyclic surfaces foliated by these curves. In this context, we obtain in Section 4 all minimal or flat cyclic surfaces foliated by geodesics, equidistant lines, or horocycles. See Theorems 4.2 and 4.4.

2 Preliminaries

The isometry group of Sol_3 has dimension 3 and the component of the identity is generated by the following families of isometries:

$$\begin{aligned} T_{1,c}(x, y, z) &:= (x + c, y, z), \\ T_{2,c}(x, y, z) &:= (x, y + c, z), \\ T_{3,c}(x, y, z) &:= (e^{-c}x, e^cy, z + c), \end{aligned}$$

where $c \in \mathbb{R}$ is a real parameter. These isometries are left multiplications by elements of Sol_3 and so, they are left-translations with respect to the structure of the Lie group. The Sol_3 space has the next three foliations:

$$\begin{aligned} \mathcal{F}_1 &: \{P_s = \{(s, y, z); y, z \in \mathbb{R}\}; s \in \mathbb{R}\} \\ \mathcal{F}_2 &: \{Q_s = \{(x, s, z); x, z \in \mathbb{R}\}; s \in \mathbb{R}\} \\ \mathcal{F}_3 &: \{R_s = \{(x, y, s); x, y \in \mathbb{R}\}; s \in \mathbb{R}\}. \end{aligned}$$

The foliations \mathcal{F}_1 and \mathcal{F}_2 are determined by the isometry groups $\{T_{1,c}; c \in \mathbb{R}\}$ and $\{T_{2,c}; c \in \mathbb{R}\}$ respectively, and they describe the (only) totally geodesic surfaces of Sol_3 , each leaf of the foliation being isometric to a hyperbolic plane. The foliations \mathcal{F}_3 are minimal surfaces, all of them being isometric to the Euclidean plane.

With respect to the metric $\langle \cdot, \cdot \rangle$, an orthonormal basis of left-invariant vector fields is given by

$$E_1 = e^{-z} \frac{\partial}{\partial x}, \quad E_2 = e^z \frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z}.$$

The Riemannian connection ∇ of Sol_3 with respect to $\{E_1, E_2, E_3\}$ is

$$\begin{aligned} \nabla_{E_1} E_1 &= -E_3 & \nabla_{E_1} E_2 &= 0 & \nabla_{E_1} E_3 &= E_1 \\ \nabla_{E_2} E_1 &= 0 & \nabla_{E_2} E_2 &= E_3 & \nabla_{E_2} E_3 &= -E_2 \\ \nabla_{E_3} E_1 &= 0 & \nabla_{E_3} E_2 &= 0 & \nabla_{E_3} E_3 &= 0. \end{aligned}$$

For a surface in Sol_3 parameterized in local coordinates by $X = X(s, t)$ and endowed with the unit normal N , the formulas of the mean curvature and Gaussian curvature are, respectively,

$$H = \frac{1}{2} \frac{En - 2Fm + Gl}{EG - F^2}, \quad K = \frac{ln - m^2}{EG - F^2},$$

with

$$\begin{aligned} E &= \langle X_s, X_s \rangle, \quad F = \langle X_s, X_t \rangle, \quad G = \langle X_t, X_t \rangle, \\ l &= \langle \nabla_{X_s} X_s, N \rangle, \quad m = \langle \nabla_{X_s} X_t, N \rangle = \langle \nabla_{X_t} X_s, N \rangle, \quad n = \langle \nabla_{X_t} X_t, N \rangle. \end{aligned}$$

Each plane P_s has constant curvature -1 and it is isometric to \mathbb{H}^2 . For an explicit isometry between both spaces, we consider \mathbb{H}^2 modeled by the upper half-plane model, that is, \mathbb{H}^2 is the upper half-plane $\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2; y > 0\}$ endowed with the metric $ds^2 = ((dx)^2 + (dy)^2)/y^2$. The isometry between P_s and \mathbb{H}^2 is given by

$$\Phi_s : P_s \rightarrow \mathbb{H}^2, \quad \Phi_s(y, z) = (y, e^z).$$

In the upper half-plane model, a circle in \mathbb{H}^2 is viewed as an Euclidean circle strictly contained in \mathbb{R}_+^2 , which it is parameterized by $\beta(t) = (a, b) + r(\cos(t), \sin(t))$, $a, b \in \mathbb{R}$, $r > 0$ and $b - r > 0$. The center of the circle β is the point $c_\beta = (a, \sqrt{b^2 - r^2})$ and the hyperbolic radius is $r_\beta = \log((b + r)/\sqrt{b^2 - r^2})$. Therefore the circle β is viewed in Sol_3 as $C_s := \Phi_s^{-1}(\beta)$ and its parametrization is then

$$\alpha_s(t) = (s, a + r \cos(t), \log(b + r \sin(t))), \quad t \in \mathbb{R}, \quad (1)$$

see Figure 1. On the other hand, we consider the foliation $\mathcal{F}_1 = \{P_s; s \in \mathbb{R}\}$ of the ambient space. The geodesics of Sol_3 intersecting orthogonally each plane of \mathcal{F}_1 are the horizontal lines of equation $y = a, z = b$, where $a, b \in \mathbb{R}$.

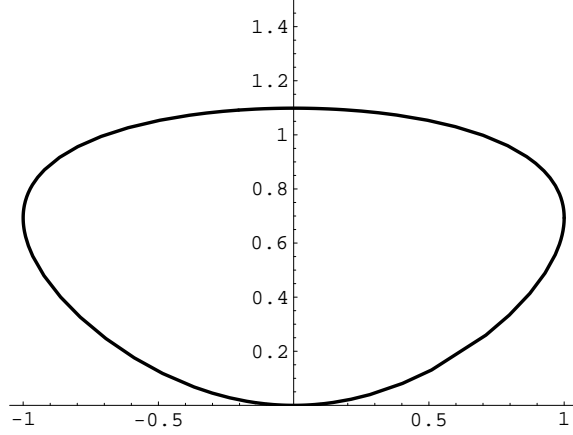


Figure 1: The circle $\Phi_s^{-1}(\beta)$ in a plane P_s , where $\beta(t) = (0, 2) + (\cos(t), \sin(t))$

A parametrization of a rotational surface in Sol_3 is

$$X(s, t) = (s, a + r(s) \cos(t), \log(b + r(s) \sin(t))),$$

where $a, b \in \mathbb{R}$, $r(s) > 0$ and $b - r > 0$. The line of centers is $s \mapsto (s, a, \sqrt{b^2 - r(s)^2})$.

We obtain a cyclic surface by considering the constants a and b as functions of the variable s . The parametrization of the surface is then

$$X(s, t) = (s, a(s) + r(s) \cos(t), \log(b(s) + r(s) \sin(t))), s \in I, t \in \mathbb{R}, \quad (2)$$

and $I \subset \mathbb{R}$ is an open interval.

Remark 2.1. *Because our results are local, we only need to compute H and K when the variable t is defined in some open interval of \mathbb{R} , that is, when we have an arc of a circle. Thus we can consider in the parametrizations (1) and (2) that $t \in J$, where $J \subset \mathbb{R}$ is an open interval. Then in the statements of the results we change 'foliated by circles' by 'foliated by arcs of circles'.*

Since in this paper we study surfaces with zero mean curvature or zero Gaussian curvature, the equations $H = 0$ and $K = 0$ are, respectively, equivalent to

$$E\langle \nabla_{X_t} X_t, \tilde{N} \rangle - 2F\langle \nabla_{X_s} X_t, \tilde{N} \rangle + G\langle \nabla_{X_s} X_s, \tilde{N} \rangle = 0, \quad (3)$$

and

$$\langle \nabla_{X_t} X_t, \tilde{N} \rangle \langle \nabla_{X_s} X_s, \tilde{N} \rangle - \langle \nabla_{X_s} X_t, \tilde{N} \rangle^2 = 0 \quad (4)$$

where \tilde{N} is a normal vector field to the surface but no necessarily unitary. It will be with both equations as we will obtain our results.

The proof of our results consists in substituting parametrization (2) into (3) and (4), obtaining a polynomial of type

$$\sum_{n=0}^k (A_n(s) \cos(nt) + B_n(s) \sin(nt)) = 0, \quad s \in I, t \in J, \quad (5)$$

for some natural number $k \in \mathbb{N}$. For each fixed $s \in I$, we view this expression as a polynomial in t ; recall that t belongs to an interval of \mathbb{R} . Because the functions $\{\cos(nt), \sin(nt); n \in \mathbb{N}\}$ are linearly independent, all coefficients $A_n(s)$ and $B_n(s)$ vanish. The explicit computations of (3) and (4) are straightforward, but complicated. We use a symbolic program as Mathematica to check these computations.

In (5) the simplest expressions correspond to the leader coefficients A_k and B_k . Using the equations $A_k = 0$ and $B_k = 0$, we get information that will be used in determining the next coefficients A_{k-1} and B_{k-1} and we repeat the process until we find our results.

3 Proof of Theorems 1.3 and 1.4

Proof. (of Theorem 1.3) Using parametrization (2) of a cyclic surface in Sol_3 we compute

$$X_s = (b(s) + r(s) \sin(t)) E_1 + \frac{a'(s) + r'(s) \cos(t)}{b(s) + r(s) \sin(t)} E_2 + \frac{b'(s) + r'(s) \sin(t)}{b(s) + r(s) \sin(t)} E_3,$$

$$X_t = \frac{r(s)}{b(s) + r(s) \sin(t)} (-\sin(t) E_2 + \cos(t) E_3),$$

and we choose the normal vector

$$\tilde{N} = \frac{r'(s) + a'(s) \cos(t) + b'(s) \sin(t)}{(b(s) + r(s) \sin(t))^2} E_1 - \cos(t) E_2 - \sin(t) E_3.$$

After some manipulations, the minimality condition yields an expression of type (5) with $k = 5$ where

$$A_5 = 0, \quad B_5 = \frac{r(s)^6}{16}.$$

Because $B_5 = 0$, we conclude $r(s) = 0$, which is a contradiction.

□

Proof. (of Theorem 1.4) Following the same reasoning as in the proof of the previous theorem, the flatness condition is equivalent to an expression of type (5) with $k = 5$ again. Here

$$A_5 = 0, \quad B_5 = \frac{1}{16}b(s)r(s)^5.$$

From $B_5 = 0$, we have $b(s) = 0$. However this is a contradiction with the fact that a circle $\Phi_s^{-1}(\beta(t))$ in P_s must satisfy the condition $b - r > 0$, see (1).

□

4 Surfaces foliated by geodesics, equidistant lines and horocycles

The curvature κ of a circle in Sol_3 satisfies $\kappa > 1$ because it is isometric to a circle of \mathbb{H}^2 . In fact, if C_s is parameterized as (1), its curvature is $\kappa_\beta = b/r$, computed with the orientation pointing inside. In the hyperbolic plane \mathbb{H}^2 , the definition of a circle is equivalent to a curve with constant curvature κ , $\kappa > 1$. However, in \mathbb{H}^2 there are other curves with constant curvature κ with $0 \leq \kappa \leq 1$. Indeed, if $\kappa = 0$, the curve is a geodesic; if $0 < \kappa < 1$, the curve is an equidistant line; and if $\kappa = 1$, the curve is a horocycle. In these cases, the curves are not closed such as it occurs for a circle. In the upper half-plane model of \mathbb{H}^2 , the three types of curves are viewed as the intersection of circles and straight-lines with \mathbb{R}_+^2 . Exactly:

1. If β is a geodesic, then β is a half-circle intersecting orthogonally the line $y = 0$ or it is a vertical straight-line $\beta(t) = (a, t)$, $a \in \mathbb{R}$, $t > 0$.
2. If β is an equidistant line, then β is the intersection of a circle of \mathbb{R}^2 with points above $y = 0$ with \mathbb{R}_+^2 , or β is a straight-line of \mathbb{R}_+^2 that intersects not orthogonally the line $y = 0$, which parametrizes as $\beta(t) = (t, at + b)$, $a, b \in \mathbb{R}$, $a \neq 0$, $t > -b/a$.
3. If β is a horocycle, then β is the intersection of a circle of \mathbb{R}^2 tangent to the line $y = 0$ with \mathbb{R}_+^2 , or β is a horizontal line of \mathbb{R}_+^2 , that is, $\beta(t) = (t, a)$, $a > 0$, $t \in \mathbb{R}$.

By the isometry ϕ_s^{-1} , we can carry all these curves to the plane P_s and we call them geodesic, equidistant line and horocycle again. Following this motivation, we generalize the concept of circle in Sol_3 assuming that the curvature of the curve is constant.

Definition 4.1. *A cyclic surface in Sol_3 by geodesics (resp. equidistant lines, horocycles) is a surface foliated by a uniparametric family of geodesics (resp. equidistant lines, horocycles) contained in the planes P_s .*

In this section, we study cyclic surfaces in Sol_3 foliated by these curves and we ask if there are minimal examples or with vanishing Gaussian curvature.

If the curve β is a part of an Euclidean circle of \mathbb{R}_+^2 , by the isometry Φ_s^{-1} this curve goes to a curve in P_s with the same expression (1) with the difference that the condition now is $b + r \sin(t) > 0$. In fact, we have that $\alpha_s = \Phi_s^{-1}(\beta)$ is geodesic if $b = 0$, an equidistant line if $0 < b < r$ and a horocycle if $0 < b = r$. See Figure 2. However, the expression of the parametrization of a cyclic surface is (2), again except the above conditions on b and r . Therefore, the same local computations as in Theorem 1.3 and 1.4 hold to compute the expressions of H and K .

From the proof of Theorem 1.3 we obtain that there are no minimal surfaces.

Concerning the flatness, from the proof of Theorem 1.4, we obtained $b(s) = 0$. In such a case, the coefficient A_4 is $A_4 = -r(s)^2 a'(s)^2 / 2$ and since A_4 must vanish, we have that $a(s)$ is a constant function $a(s) = a_0$, $a_0 \in \mathbb{R}$. Furthermore, $A_2 = -8r(s)^2 r'(s)^2$, concluding that $r(s)$ is a constant function too.

These results for surfaces parameterized by (2) will appear in the statements of Theorems 4.2 and 4.4.

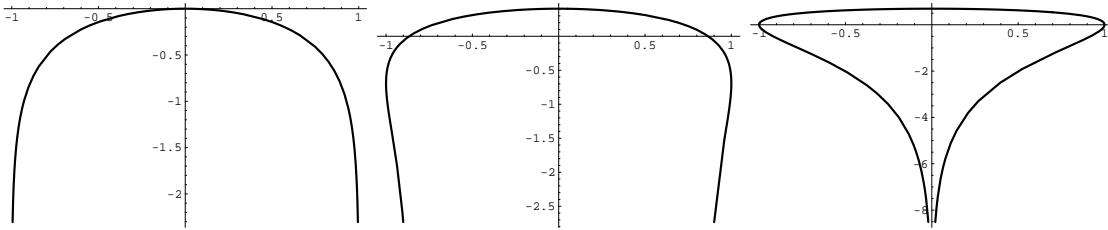


Figure 2: A geodesic, an equidistant line and a horocycle in a plane P_s . The curves are obtained by $\Phi_s^{-1}(\beta)$ and $\beta(t) = (a, b) + r(\cos(t), \sin(t))$. For a geodesic, $b = 0$; for an equidistant line, $0 < b < r$; and for a horocycle, $b = r > 0$.

As a consequence, the only cases that we have to study are when the curve β is a straight-line in \mathbb{H}^2 . By the isometry Φ_s^{-1} , these curves transform into:

1. If β is a geodesic, $\Phi_s^{-1}(\beta(t)) = (s, a, \log(t)), t > 0$.
2. If β is a equidistant line, $\Phi_s^{-1}(\beta(t)) = (s, t, \log(at + b))$.
3. If β is a horocycle, $\Phi_s^{-1}(\beta(t)) = (s, t, \log(a)), a > 0$.

Using these parametrizations, we prove:

Theorem 4.2. *Consider a cyclic surface in Sol_3 with $H = 0$.*

1. *If it is foliated by geodesics, the surface is a vertical plane $X(s, t) = (s, \lambda s + \mu, t)$, $\lambda, \mu \in \mathbb{R}$.*
2. *If it is foliated by equidistant lines, the surface is $X(s, t) = \left(s, t, \log \left| \frac{b_0}{s + b_1} \right| \right)$ or $X(s, t) = \left(s, t, \log \left| \frac{a_0 t}{s + a_1} \right| \right)$, where $a_0, a_1, b_0, b_1 \in \mathbb{R}$.*
3. *If it is foliated by horocycles, the surface is $X(s, t) = (s, t, \lambda + \log |s + \mu|)$, $\lambda, \mu \in \mathbb{R}$.*

Proof. We distinguish the three cases.

1. The surface is foliated by geodesics. The parametrization of the surface is given by $X(s, t) = (s, a(s), t)$, for some smooth function $a(s)$, $t \in \mathbb{R}$. Now we have

$$X_s = e^t E_1 + a'(s) e^{-t} E_2, \quad X_t = E_3, \quad \tilde{N} = a'(s) e^{-t} E_1 - e^t E_2.$$

The minimality condition is equivalent to $a''(s) = 0$. Thus $a(s) = \lambda s + \mu$ for some constants $\lambda, \mu \in \mathbb{R}$. In this case, the surface is a vertical plane.

2. The surface is foliated by equidistant lines. The parametrization of the surface is $X(s, t) = (s, t, \log(a(s)t + b(s)))$, with $a(s)t + b(s) > 0$. Here

$$X_s = (a(s)t + b(s)) E_1 + \frac{a'(s)t + b'(s)}{a(s)t + b(s)} E_3, \quad X_t = \frac{E_2 + a(s) E_3}{a(s)t + b(s)},$$

$$\tilde{N} = -\frac{a'(s)t + b'(s)}{(a(s)t + b(s))^2} E_1 - a(s) E_2 + E_3.$$

We compute the equation $H = 0$ obtaining a 2-degree polynomial in t in the form $\sum_{n=0}^2 A_n(s)t^n = 0$. Therefore, the coefficients $A_n(s)$ vanish on its domain. The vanishing coefficient $A_2 = (1 + a(s)^2)(-2a'(s)^2 + a(s)a''(s))$ yields $a(s) = a_0/(s + a_1)$, for some constants $a_0, a_1 \in \mathbb{R}$. Using this information we get

$$A_1 = \frac{a_0}{(a_1+s)^4} (2(a_1+s)b(s) + 2(2a_1^2 + a_0^2 + 4a_1s + 2s^2)b'(s) + (a_1+s)(a_1^2 + a_0^2 + 2a_1s + s^2)b''(s)).$$

When A_1 vanishes identically, $a_0 = 0$ or $b(s) = 0$. The first case yields $a(s) = 0$. Then, the minimality condition $H = 0$ writes as $-2b'(s)^2 + b(s)b''(s) = 0$, with the solution $b(s) = b_0/(s + b_1)$, $b_0, b_1 \in \mathbb{R}$.

In the second case, if $b(s) = 0$ then the surface is always minimal because now the expression $H = 0$ is trivial.

3. The surface is foliated by horocycles. The parametrization of the surface is given by $X(s, t) = (s, t, a(s))$, where $a(s)$ is a smooth function. We compute

$$X_s = e^{a(s)}E_1 + a'(s)E_3, \quad X_t = e^{-a(s)}E_2, \quad \tilde{N} = a'(s)e^{-a(s)}E_1 - E_3.$$

Here the condition $H = 0$ is equivalent to $a''(s) - a'(s)^2 = 0$, and the solution is $a(s) = \lambda + \log |s + \mu|$, with $\lambda, \mu \in \mathbb{R}$.

□

Remark 4.3. *Part of the surfaces obtained in Theorem 4.2 are invariant by the families $\{T_{1,c}\}$ or $\{T_{2,c}\}$. Indeed, the first surface foliated by equidistant lines is T_2 -invariant and the second one is T_1 -invariant. The surfaces foliated by horocycles are T_2 -invariant. All these surfaces appeared in [7].*

In a similar manner, we study flat cyclic surfaces foliated by geodesics, equidistant lines or horocycles.

Theorem 4.4. *Consider a cyclic surface in Sol_3 with $K = 0$.*

1. *If it is foliated by geodesics, then the surface is given by*

(1.a) *a plane Q_s , $s \in \mathbb{R}$,*

(1.b) *or $X(s, t) = (s, a + r \cos(t), \log(r \sin(t)))$, with $a, r \in \mathbb{R}$, $r > 0$.*

2. If it is foliated by equidistant lines, then the surface is

$$X(s, t) = \left(s, t, \log \left| \frac{1}{\sqrt{-s^2 + \lambda s + \mu}} \right| \right), \lambda, \mu \in \mathbb{R}.$$

3. If it is foliated by horocycles, then the parametrization of the surface is

$$X(s, t) = \left(s, t, -\frac{1}{2} \log |-s^2 + \lambda s + \mu| \right), \lambda, \mu \in \mathbb{R}.$$

Proof. Following the same steps as in the proof of the previous theorem, we discuss the three cases.

1. The condition $K = 0$ for a cyclic surface foliated by geodesics is equivalent to $a'(s)^2 = 0$, which yields $a(s) = a_0$. Thus, case (1.a) of the first item of the theorem is proved.

Case (1.b) corresponds to a flat surface parameterized by (2).

2. The flatness condition for surfaces foliated by equidistant lines is equivalent to a vanishing 4-degree polynomial, $\sum_{n=0}^4 A_n(s)t^n = 0$. Thus, all the coefficients must be zero. The leader coefficient is $A_4 = -a(s)^4(1 + a(s)^2)$, which implies $a(s) = 0$. Now $K = 0$ simply writes as $b(s)^4 + 3b'(s)^2 - b(s)b''(s) = 0$. The solution is $b(s) = \frac{\pm 1}{\sqrt{-s^2 + \lambda s + \mu}}$, and it yields the parametrization from the second item of the theorem.

3. The condition $K = 0$ for a cyclic surface foliated by horocycles is equivalent to $a''(s) - 2a'(s)^2 - e^{2a(s)} = 0$. The solution is given by $a(s) = -\frac{1}{2} \log |-s^2 + \lambda s + \mu|$, $\lambda, \mu \in \mathbb{R}$, and the last item of the theorem is proved.

□

The surfaces foliated by equidistant lines and by horocycles obtained in Theorem 4.4 are now T_2 -invariant. On the other hand, the surface foliated by geodesics is T_1 -invariant. Again, these surfaces appeared in [7].

Corollary 4.5. *Any minimal or flat cyclic surface in Sol_3 foliated by geodesics, equidistant lines or horocycles is invariant by the family of translations $\{T_{1,c}\}_{c \in \mathbb{R}}$ or $\{T_{2,c}\}_{c \in \mathbb{R}}$.*

5 Surfaces foliated by circles in the z -planes

The space Sol_3 is foliated by the surfaces R_s of equation $z = s$. Each surface R_s is isometric to the Euclidean plane \mathbb{E}^2 and they are minimal surfaces in Sol_3 . We consider in R_s curves equidistant from a fixed point of R_s . We call these curves *circles* in R_s . Here the distance is computed as the intrinsic distance on the surface R_s . The induced metric on R_s is $e^{2s}(dx)^2 + e^{-2s}(dy)^2$ and an isometry $\Psi : R_s \rightarrow \mathbb{E}^2$ is $\Psi(s, x, y) = (e^s x, e^{-s} y)$. Therefore, a circle in R_s is the inverse image of an Euclidean circle of \mathbb{E}^2 and this circle is parametrized as $\alpha_s(t) = (a + re^{-s} \cos(t), b + re^s \sin(t), s)$. Thus we can consider surfaces in Sol_3 foliated by circles at the horizontal planes R_s , which will be called cyclic surfaces. Again, we ask for those surfaces with zero mean curvature or zero Gaussian curvature. A local parametrization of such a surface is

$$X(s, t) = (a(s) + r(s)e^{-s} \cos(t), b(s) + r(s)e^s \sin(t), s), \quad s \in I, t \in J, \quad (6)$$

where $I, J \subset \mathbb{R}$ are open intervals.

Theorem 5.1. *There are no minimal or flat cyclic surfaces foliated by a uniparametric family of circles in the planes R_s .*

Proof. A parametrization of such a cyclic surface is given by (6). We have

$$\begin{aligned} X_s &= (a'e^s + (r' - r) \cos(t))E_1 + (b'e^{-s} + (r' + r) \sin(t))E_2 + E_3, \\ X_t &= -r \sin(t)E_1 + r \cos(t)E_2. \end{aligned}$$

We choose the normal vector to the surface

$$\tilde{N} = \cos(t)E_1 + \sin(t)E_2 + (r \cos(2t) - a'e^s \cos(t) - b'e^{-s} \sin(t) - r')E_3.$$

1. Assume that the surface is minimal. The equation (3) writes as an expression of type (5) with $k = 4$. In fact, $A_4 = -r(s)^3/2$ and $B_4 = 0$. Because $A_4 = 0$, then $r = 0$, obtaining a contradiction.
2. If the surface is flat, the expression (4) is a polynomial equation (5) with $k = 8$. Here $A_8 = -r(s)^6/8$ and $B_8 = 0$. Again, we obtain a contradiction.

□

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